Free field realization of commutative family of elliptic Feigin-Odesskii algebra

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Abstract

In this review, we study free field realizations of the Feigin-Odesskii algebra. We construct free field realizations of a pair of infinitely many commutative operators, associated with the elliptic algebra $U_{q,p}(\widehat{sl_N})$.

1 Introduction

In this review, we study free field realization of elliptic version of the Feigin-Odesskii algebra [1]. For this purpose we introduce one parameter "s" deformation of the Feigin-Odesskii algebra [1]. This review is based on the paper [9, 10, 11, 16, 17]. Let the function $f_l(z_1 \cdots z_l | w_1 \cdots w_l)$ be meromorphic and symmetric in each of varibles $(z_1, \cdots, z_l)$ and $(w_1, \cdots, w_l)$. Let us set the symmetric function $(f_m \circ f_n)(z_1, \cdots, z_{m+n} | w_1, \cdots, w_{m+n})$, depending on three continuous parameters $0 < x < 1, 0 < r$ and $0 < s < 2$, by

$$\begin{aligned} & (f_m \circ f_n)(z_1, \cdots, z_{m+n} | w_1, \cdots, w_{m+n}) \\ = & \frac{1}{((m+n)!)^2} \sum_{\sigma \in S_{m+n}} \sum_{\tau \in S_{m+n}} f_m(z_{\sigma(1)}, \cdots, z_{\sigma(m)} | w_{\tau(1)}, \cdots, w_{\tau(m)}) \\ & \times f_n(z_{\sigma(m+1)}, \cdots, z_{\sigma(m+n)} | w_{\tau(m+1)}, \cdots, w_{\tau(m+n)}) \\ & \times \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[v_{\tau(i)} - u_{\sigma(j)} + \frac{s}{2}\right]_r \left[u_{\sigma(i)} - v_{\tau(j)} + \frac{s}{2}\right]_r}{[u_{\sigma(i)} - u_{\sigma(j)}]_r [u_{\sigma(j)} - u_{\sigma(i)} - 1]_r}. \\ & \times \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[u_{\sigma(j)} - v_{\tau(i)} + \frac{s}{2} - 1\right]_r \left[v_{\tau(j)} - u_{\sigma(i)} + \frac{s}{2} - 1\right]_r}{[v_{\tau(j)} - v_{\tau(i)} - 1]_r [v_{\tau(j)} - v_{\tau(i)} - 1]_r}, \end{aligned}$$

where the symbol $[u]_r$ represents the elliptic theta function defined in (2.1). Here we set $z_j = x^{2u_j}, w_j = x^{2v_j}$. This product "$\circ$" on symmetric function gives the structure of the associative

algebra. We call this associative algebra "elliptic Feigin-Odesskii algebra". Let us set the functional $\mathcal{G}$ by using currents $F_1(z), F_2(z)$, which is one parameter "s" deformation of the elliptic algebra $U_{q,p}(\widehat{sl_2})$. They satisfy the following commutation relations.

$$\begin{aligned}\left[u_1-u_2-\frac{s}{2}\right]_r\left[u_1-u_2+\frac{s}{2}-1\right]_r F_1(z_1)F_2(z_2) &= \left[u_2-u_1-\frac{s}{2}\right]_r\left[u_2-u_1+\frac{s}{2}-1\right]_r F_1(z_2)F_2(z_1),\\ [u_1-u_2]_r[u_1-u_2+1]_r F_j(z_1)F_j(z_2) &= [u_2-u_1]_r[u_2-u_1+1]_r F_j(z_2)F_j(z_1).\end{aligned}$$

Upon the specialization $s\to 2$ the current $F_1(z)$ degenerates to the cuurent of the elliptic algebra $U_{q,p}(\widehat{sl_2})$, and the current $F_2(z)$ lookes like $F_1(z)^{-1}$. Let us set the functional $\mathcal{G}$ by

$$\begin{aligned}\mathcal{G}(f_m) &= \oint\prod_{j=1}^m\frac{dz_j}{2\pi i z_j}\oint\prod_{j=1}^m\frac{dw_j}{2\pi i w_j}F_1(z_1)\cdots F_1(z_m)F_2(w_1)\cdots F_2(w_m)\\ &\times\frac{\displaystyle\prod_{1\leqq j<k\leqq m}[u_i-u_j]_r[u_j-u_i-1]_r[v_i-v_j]_r[v_j-v_i-1]_r}{\displaystyle\prod_{i=1}^m\prod_{j=1}^m\left[u_i-v_j+\frac{s}{2}\right]_r\left[v_j-u_i+\frac{s}{2}-1\right]_r}f_m(z_1,\cdots,z_m|w_1,\cdots,w_m).\end{aligned}$$

Roghly speaking, this functional satisfies homomorphism,

$$\mathcal{G}(f_m)\mathcal{G}(f_n)=\mathcal{G}(f_m\circ f_n),$$

which is a consequence of symmetrizing procedure of variables $(z_1,\cdots,z_{m+n})$ and $(w_1,\cdots,w_{m+n})$. We call $\mathcal{G}(f_m)$ "free field realization of Feigin-Odesskii algebra". When we have commutative family ϑ_m of elliptic Feigin-Odesskii algebra,

$$\vartheta_m\circ\vartheta_n=\vartheta_n\circ\vartheta_m,$$

we can construct commutative family of the operators $\mathcal{G}(\vartheta_m)$,

$$\mathcal{G}(\vartheta_m)\cdot\mathcal{G}(\vartheta_n)=\mathcal{G}(\vartheta_n)\cdot\mathcal{G}(\vartheta_m).$$

This is rough story of this paper. Precisely this homomorphism $\mathcal{G}(f_m)\mathcal{G}(f_n)=\mathcal{G}(f_m\circ f_n)$ does not hold for every time. For example, upon the specialization $s\to 2$, the homomorphism does not hold, because singularity which comes from the product of the current $E_1(z), E_2(w)$, destroy the structure. In order to construct the free field realization of Feigin-Odesskii algebra, we have to construct the currents which (1) satisfy the commutation relation, and (2) does not have surplus singularity. In this survey we construct free field realization of commuttive family of elliptic Feigin-Odesskii algebra.

The organization of this paper is as follows. In section 2 we introduce a pair of Feigin-Odesskii algebra, and give infinitely many commutative solutions of the Feigin-Odesskii algebra. We

construct free field realization of the Feigin-Odesskii algebra, by using one parameter deformation of the current of the elliptic algebra $U_{q,p}(\widehat{sl_2})$. In terms of this free field realization, we construct a pair of infinitely many commutative operators acting on the Fock space. In section 3 we consider the higher-rank generalization of section 2. We construct a pair infinitely many commutative operators by using one parameter deformation of the elliptic algebra $U_{q,p}(\widehat{sl_N})$. In section 4 we consider higher-level k generalization of section 2. We construct free field realization of one parameter deformation of level k elliptic algebra $U_{q,p}(\widehat{sl_2})$. The author would like to emphasize that the free field realization of Level k is completely different from those of Level 1. We construct a pair of infinitely many commutative operators associated with one parameter s deformation of the elliptic algebra $U_{q,p}(\widehat{sl_2})$ for level k. In section 5 we give a free field realization of the elliptic algebra $U_{q,p}(\widehat{sl_N})$ for level k, and explain an open problem. In section 2 we summarize some of results in [9]. In section 3 we summarize some of results in [10]. In section 4 we summarize the results in [11]. In section 5 we summarize the results in [16, 17].

2 Elliptic algebra $U_{q,p}(\widehat{sl_2})$

Let us fix parameters $0 < x < 1$, $r > 0$. Let us set $z = x^{2u}$. The symbol $[u]_r$ stands for the Jacobi theta function,

$$[u]_r = x^{\frac{u^2}{r}-u}\frac{\Theta_{x^{2r}}(z)}{(x^{2r};x^{2r})_\infty^3}, \quad \Theta_q(z) = (q;q)_\infty(z;q)_\infty(q/z;q)_\infty, \tag{2.1}$$

where we have used standard notation $(z;q)_\infty = \prod_{j=0}^\infty(1-q^jz)$. The symbol $[a]$ stands for q-integer,

$$[a] = \frac{x^a - x^{-a}}{x - x^{-1}}. \tag{2.2}$$

2.1 Feigin-Odesskii algebra

Let us set parameters $0 < s < 2$ and $r > 1$. We introduce a pair of Feigin-Odesskii algebra: $f \circ g$ and $f * g$.

Definition 2.1 *Let us set the symmetric function* $(f_m \circ f_n)(z_1, \cdots, z_{m+n}|w_1, \cdots, w_{m+n})$ *by*

$$\begin{aligned}&(f_m \circ f_n)(z_1, \cdots, z_{m+n}|w_1, \cdots, w_{m+n})\\ =\ &\frac{1}{((m+n)!)^2}\sum_{\sigma\in S_{m+n}}\sum_{\tau\in S_{m+n}} f_m(z_{\sigma(1)}, \cdots, z_{\sigma(m)}|w_{\tau(1)}, \cdots, w_{\tau(m)})\\ &\times f_n(z_{\sigma(m+1)}, \cdots, z_{\sigma(m+n)}|w_{\tau(m+1)}, \cdots, w_{\tau(m+n)})\end{aligned}$$

$$\times\prod_{i=1}^{m}\prod_{j=m+1}^{m+n}\frac{\left[v_{\tau(i)}-u_{\sigma(j)}+\frac{s}{2}\right]_r\left[u_{\sigma(i)}-v_{\tau(j)}+\frac{s}{2}\right]_r}{[u_{\sigma(i)}-u_{\sigma(j)}]_r[u_{\sigma(j)}-u_{\sigma(i)}-1]_r}$$

$$\times\ \prod_{i=1}^{m}\prod_{j=m+1}^{m+n}\frac{\left[u_{\sigma(j)}-v_{\tau(i)}+\frac{s}{2}-1\right]_r\left[v_{\tau(j)}-u_{\sigma(i)}+\frac{s}{2}-1\right]_r}{[v_{\tau(j)}-v_{\tau(i)}-1]_r[v_{\tau(j)}-v_{\tau(i)}-1]_r}. \tag{2.3}$$

Let us give the symmetric function $(f_m * f_n)(z_1\cdots z_{m+n}|w_1\cdots w_{m+n})$ *by*

$$\begin{aligned}
&(f_m * f_n)(z_1,\cdots,z_{m+n}|w_1,\cdots,w_{m+n})\\
=\ &\frac{1}{((m+n)!)^2}\sum_{\sigma\in S_{m+n}}\sum_{\tau\in S_{m+n}}f_m(z_{\sigma(1)},\cdots,z_{\sigma(m)}|w_{\tau(1)},\cdots,w_{\tau(m)})\\
&\times f_n(z_{\sigma(m+1)},\cdots,z_{\sigma(m+n)}|w_{\tau(m+1)},\cdots,w_{\tau(m+n)})\\
&\times\prod_{i=1}^{m}\prod_{j=m+1}^{m+n}\frac{\left[v_{\tau(i)}-u_{\sigma(j)}-\frac{s}{2}\right]_{r-1}\left[u_{\sigma(i)}-v_{\tau(j)}-\frac{s}{2}\right]_{r-1}}{[u_{\sigma(i)}-u_{\sigma(j)}]_{r-1}[u_{\sigma(j)}-u_{\sigma(i)}+1]_{r-1}}\\
\times\ &\prod_{i=1}^{m}\prod_{j=m+1}^{m+n}\frac{\left[u_{\sigma(j)}-v_{\tau(i)}-\frac{s}{2}+1\right]_{r-1}\left[v_{\tau(j)}-u_{\sigma(i)}-\frac{s}{2}+1\right]_{r-1}}{[v_{\tau(j)}-v_{\tau(i)}+1]_{r-1}[v_{\tau(j)}-v_{\tau(i)}+1]_{r-1}}.
\end{aligned} \tag{2.4}$$

Here $f_l(z_1,\cdots,z_l|w_1,\cdots,w_l)$ *are meromorphic function symmetric in each of varibles* $(z_1,\cdots,z_l)$ *and* $(w_1,\cdots,w_l)$.

We have infinitely mny commutative family of Feigin-Odesskii algebra. Let us set theta functions for three parameters α,ν,

$$\vartheta_{m,\alpha}(z_1,\cdots,z_m|w_1,\cdots,w_m)=\left[\sum_{j=1}^{m}(u_j-v_j)-\nu+\alpha\right]_r\left[\sum_{j=1}^{m}(v_j-u_j)-\alpha\right]_r. \tag{2.5}$$

Proposition 2.2 *$\vartheta_{m,\alpha}$ and $\vartheta_{n,\beta}$ commute with respect to the product (2.3).*

$$\vartheta_{m,\alpha}\circ\vartheta_{n,\beta}=\vartheta_{n,\beta}\circ\vartheta_{m,\alpha}. \tag{2.6}$$

Let us set theta functions for parmeters α,ν.

$$\vartheta^*_{m,\alpha}(z_1,\cdots,z_m|w_1,\cdots,w_m)=\left[\sum_{j=1}^{m}(v_j-u_j)-\nu+\alpha\right]_{r-1}\left[\sum_{j=1}^{m}(u_j-v_j)-\alpha\right]_{r-1}. \tag{2.7}$$

Proposition 2.3 *$\vartheta^*_{m,\alpha}$ and $\vartheta^*_{n,\beta}$ commute with respect to the product (2.4).*

$$\vartheta^*_{m,\alpha}*\vartheta^*_{n,\beta}=\vartheta^*_{n,\beta}*\vartheta^*_{m,\alpha}. \tag{2.8}$$

Proof of propositions are summarized in [9].

2.2 Free field realization

Let us set a parameter $0 < s < 2$. Let us introduce bosons β_m^1, β_m^2, $(m \neq 0)$ by

$$[\beta_m^i, \beta_n^j] = \begin{cases} m\frac{[(r-1)m]}{[rm]}\frac{[(s-1)m]}{[sm]}\delta_{m+n,0}, & (i=j) \\ -m\frac{[(r-1)m]}{[rm]}\frac{[m]}{[sm]}x^{sm\ \mathrm{sgn}(i-j)}\delta_{m,n}, & (i \neq j) \end{cases} \tag{2.9}$$

Let us set P, Q by

$$[P, iQ] = 1. \tag{2.10}$$

We deal with the bosonic Fock space $\mathcal{F}_{l,k}$,$(l,k \in \mathbb{Z})$ generated by β_{-m}^i,$(m>0, i=1,2)$ over the vacuum vector $|l,k\rangle$.

$$\beta_m^i|l,k\rangle = 0\ (m>0, i=1,2), \tag{2.11}$$

$$P|l,k\rangle = \left(\sqrt{\frac{r}{2(r-1)}}l - \sqrt{\frac{r-1}{2r}}k\right)|l,k\rangle, \tag{2.12}$$

$$|l,k\rangle = e^{\left(\sqrt{\frac{r}{2(r-1)}}l - \sqrt{\frac{r-1}{2r}}k\right)iQ}|0,0\rangle. \tag{2.13}$$

Definition 2.4 *Let us set the currents $F_j(z), E_j(z)$,$(j=1,2)$ by*

$$F_1(z) = z^{\frac{r-1}{r}}e^{i\sqrt{\frac{2(r-1)}{r}}Q}z^{\sqrt{\frac{2(r-1)}{r}}P} : \exp\left(\sum_{m\neq 0}\frac{1}{m}(\beta_m^1 - \beta_m^2)z^{-m}\right) :, \tag{2.14}$$

$$F_2(z) = z^{\frac{r-1}{r}}e^{-i\sqrt{\frac{2(r-1)}{r}}Q}z^{-\sqrt{\frac{2(r-1)}{r}}P} : \exp\left(\sum_{m\neq 0}\frac{1}{m}(-x^{sm}\beta_m^1 + x^{-sm}\beta_m^2)z^{-m}\right) :, \tag{2.15}$$

$$E_1(z) = z^{\frac{r}{r-1}}e^{-i\sqrt{\frac{2r}{r-1}}Q}z^{-\sqrt{\frac{2r}{r-1}}P} : \exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]}{[(r-1)m]}(\beta_m^1 - \beta_m^2)z^{-m}\right) :, \tag{2.16}$$

$$E_2(z) = z^{\frac{r}{r-1}}e^{i\sqrt{\frac{2r}{r-1}}Q}z^{\sqrt{\frac{2r}{r-1}}P} : \exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]_x}{[(r-1)m]_x}(-x^{sm}\beta_m^1 + x^{-sm}\beta_m^2)z^{-m}\right) \tag{2.17}$$

They satisfy the following commutation relations.

Proposition 2.5

$$\frac{[u_1-u_2]_r}{[u_1-u_2-1]_r}F_j(z_1)F_j(z_2) = \frac{[u_2-u_1]_r}{[u_2-u_1-1]_r}F_j(z_2)F_j(z_1), \quad (j=1,2) \tag{2.18}$$

$$\frac{[u_1-u_2+\frac{s}{2}-1]_r}{[u_1-u_2+\frac{s}{2}]_r}F_1(z_1)F_2(z_2) = \frac{[u_2-u_1+\frac{s}{2}-1]_r}{[u_2-u_1+\frac{s}{2}]_r}F_2(z_2)F_1(z_1), \tag{2.19}$$

$$\frac{[u_1-u_2]_{r-1}}{[u_1-u_2+1]_{r-1}}E_j(z_1)E_j(z_2) = \frac{[u_2-u_1]_{r-1}}{[u_2-u_1+1]_{r-1}}E_j(z_2)E_j(z_1), \quad (j=1,2) \tag{2.20}$$

$$\frac{[u_1-u_2-\frac{s}{2}+1]_{r-1}}{[u_1-u_2-\frac{s}{2}]_{r-1}}E_1(z_1)E_2(z_2) = \frac{[u_2-u_1-\frac{s}{2}+1]_{r-1}}{[u_2-u_1-\frac{s}{2}]_{r-1}}E_2(z_2)E_1(z_1). \tag{2.21}$$

$$
\begin{aligned}
&[E_i(z_1),F_j(z_2)]\\
=\ &\frac{\delta_{i,j}}{x-x^{-1}}\left(\delta(xz_2/z_1)H_j(x^rz_2)-\delta(xz_1/z_2)H_j(x^{-r}z_2)\right),\quad (i,j=1,2). \qquad (2.22)
\end{aligned}
$$

Here we have set

$$H_1(z) = e^{-\frac{1}{\sqrt{r(r-1)}}iQ}z^{-\frac{1}{\sqrt{r(r-1)}}P+\frac{1}{r(r-1)}} :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]}{[(r-1)m]}(\beta_m^1-\beta_m^2)z^{-m}\right):, \qquad (2.23)$$

$$H_2(z) = e^{\frac{1}{\sqrt{r(r-1)}}iQ}z^{\frac{1}{\sqrt{r(r-1)}}P+\frac{1}{r(r-1)}} :\exp\left(\sum_{m\neq 0}\frac{1}{m}\frac{[m]}{[(r-1)m]}(x^{sm}\beta_m^1-x^{-sm}\beta_m^2)z^{-m}\right) \qquad (2.24)$$

Definition 2.6 *Let us set the functional $\mathcal{G}$ by*

$$
\begin{aligned}
\mathcal{G}(f_m) =& \oint\prod_{j=1}^m\frac{dz_j}{2\pi i z_j}\oint\prod_{j=1}^m\frac{dw_j}{2\pi i w_j}F_1(z_1)\cdots F_1(z_m)F_2(w_1)\cdots F_2(w_m) \qquad (2.25)\\
&\times\frac{\prod_{1\leqq j<k\leqq m}[u_i-u_j]_r[u_j-u_i-1]_r[v_i-v_j]_r[v_j-v_i-1]_r}{\prod_{i=1}^m\prod_{j=1}^m\left[u_i-v_j+\frac{s}{2}\right]_r\left[v_j-u_i-\frac{s}{2}+1\right]_{r^*}}f_m(z_1,\cdots,z_m|w_1,\cdots,w_m).
\end{aligned}
$$

We take the integration contours to be simple closed curves around the origin satisfying

$$|x^sw_i|,|x^{2-s}w_i|<|z_j|<|x^{-s}w_i|,|x^{s-2}w_i|,\quad (i,j=1,2,\cdots,m).$$

Let us set the functional $\mathcal{G}^*$ by followings.

$$
\begin{aligned}
\mathcal{G}^*(f_m) =& \oint\prod_{j=1}^m\frac{dz_j}{2\pi i z_j}\oint\prod_{j=1}^m\frac{dw_j}{2\pi i w_j}E_1(z_1)\cdots E_1(z_m)E_2(w_1)\cdots E_2(w_m) \qquad (2.26)\\
&\times\frac{\prod_{1\leqq j<k\leqq m}[u_i-u_j]_{r-1}[u_j-u_i+1]_{r-1}[v_i-v_j]_{r-1}[v_j-v_i+1]_{r-1}}{\prod_{i=1}^m\prod_{j=1}^m\left[u_i-v_j-\frac{s}{2}\right]_{r-1}\left[v_j-u_i-\frac{s}{2}+1\right]_{r-1}}f_m(z_1,\cdots,z_m|w_1,\cdots,w_m).
\end{aligned}
$$

We take the integration contours to be simple closed curves around the origin satisfying

$$|x^sw_i|,|x^{2-s}w_i|<|z_j|<|x^{-s}w_i|,|x^{s-2}w_i|,\quad (i,j=1,2,\cdots,m).$$

Proposition 2.7 *When the function $f_l(z_1,\cdots,z_l|w_1,\cdots,w_l)$ are meromorphic function symmetric in each of varibles $(z_1,\cdots,z_l)$, $(w_1,\cdots,w_l)$, and don't have poles at the origin $z_j=0$, $w_j=0$, the functionals $\mathcal{G}$, $\mathcal{G}^*$ satisfy*

$$\mathcal{G}(f_m)\mathcal{G}(f_n)=\mathcal{G}(f_m\circ f_n), \qquad (2.27)$$

$$\mathcal{G}^*(f_m)\mathcal{G}^*(f_n)=\mathcal{G}^*(f_m * f_n). \qquad (2.28)$$

Symmetrizing with respect to the integration variables $(z_1, \cdots, z_{m+n})$, $(w_1, \cdots, w_{m+n})$ of product $\mathcal{G}(f_m)\mathcal{G}(f_n)$, we have the above proposition. We have to choose the integration contours symmetric with respect with integration variables. Hence, following normal orderings,

$$F_1(z)F_2(w) =:: x^{-\frac{2(r-1)}{r}} \frac{(x^{2r-2+s}w/z;x^{2r})_\infty (x^{2r-s}w/z)_\infty}{(x^s w/z;x^{2r})_\infty (x^{2-s}w/z;x^{2r})_\infty},$$
$$F_2(w)F_1(z) =:: x^{-\frac{2(r-1)}{r}} \frac{(x^{2r-2+s}z/w;x^{2r})_\infty (x^{2r-s}z/w)_\infty}{(x^s z/w;x^{2r})_\infty (x^{2-s}z/w;x^{2r})_\infty}.$$

we have to choose the integration contours to be simple closed curves around the origin satisfying

$$|x^s w_i|, |x^{2-s}w_i| < |z_j| < |x^{-s}w_i|, |x^{s-2}w_i|, \quad (i,j = 1,2,\cdots,m).$$

When we consider the case $s \to 2$ or $s \to 0$, there does not exist such a contour. Hence the above proposition does not hold. We note that one deformation parameter $0 < s < 2$ plays an essential role in construction of commutative operators. When we take the limit $s \to 2$, we get popular current of the elliptic algebra $U_{q,p}(\widehat{sl_2})$, and the free field realization of Feigin-Odesskii algebra is open problem. In what follows, we set $\nu = \sqrt{r(r-1)}P$.

Theorem 2.8 *For $r > 1$ we have*

$$[\mathcal{G}(\vartheta_{m,\alpha}), \mathcal{G}(\vartheta_{n,\beta})] = 0, \quad (m,n \in \mathbb{N}), \tag{2.29}$$
$$[\mathcal{G}^*(\vartheta^*_{m,\alpha}), \mathcal{G}^*(\vartheta^*_{n,\beta})] = 0, \quad (m,n \in \mathbb{N}). \tag{2.30}$$

Theorem 2.9 *For $0 < r < 1$ we have*

$$[\mathcal{G}(\vartheta_{m,\alpha}), \mathcal{G}^*(\vartheta^*_{n,\beta})] = 0, \quad (m,n \in \mathbb{N}). \tag{2.31}$$

Definition of $\mathcal{G}^*(\vartheta^*_{m,\alpha})$ for $0 < r < 1$ is given as the same manner as (4.44). See detailds in [9]. We have constructed infinitely many commutative operators $\mathcal{G}(\vartheta_{m,\alpha})$, $\mathcal{G}^*(\vartheta^*_{m,\alpha})$, $(m \in \mathbb{N})$ acting on the bosonic Fock space, which is regarded as the free field realization of commutative family of Feigin-Odesskii algebra (2.3) and (2.4).

3 Elliptic algebra $U_{q,p}(\widehat{sl_N})$

In this section we summarize some of results in [10]. In this section we fix $N = 3, 4, \cdots$. We set parameters $0 < s < N$.

3.1 Feigin-Odesskii algebra

We introduce a pair of Feigin-Odesskii algebra. We set $z_j^{(t)}=x^{2u_j^{(t)}}$ and understand $z_j^{(t+N)}=z_j^{(t)}$.

Definition 3.1 *Let us set meromorphic function* $(f_m\circ f_n)(z_1^{(1)},\cdots,z_{m+n}^{(1)}|\cdots|z_1^{(N)},\cdots,z_{m+n}^{(N)})$ *symmetric in each of variables* $(z_1^{(1)},\cdots,z_{m+n}^{(1)}),\cdots$ $(z_1^{(N)},\cdots,z_{m+n}^{(N)})$.

$$\begin{aligned}
&(f_m\circ f_n)(z_1^{(1)},\cdots,z_{m+n}^{(1)}|\cdots|z_1^{(N)},\cdots,z_{m+n}^{(N)})\\
=&\sum_{\sigma_1\in S_{m+n}}\sum_{\sigma_2\in S_{m+n}}\cdots\sum_{\sigma_N\in S_{m+n}}\\
\times&\ f_m(z_{\sigma_1(1)}^{(1)},\cdots,z_{\sigma_1(m)}^{(1)}|\cdots|z_{\sigma_N(1)}^{(N)},\cdots,z_{\sigma_N(m)}^{(N)})\\
\times&\ f_n(z_{\sigma_1(m+1)}^{(1)},\cdots,z_{\sigma_1(m+n)}^{(1)}|\cdots|z_{\sigma_N(m+1)}^{(N)},\cdots,z_{\sigma_N(m+n)}^{(N)})\\
\times&\prod_{t=1}^N\prod_{i=1}^m\prod_{j=m+1}^{m+n}\frac{\left[u_{\sigma_t(i)}^{(t)}-u_{\sigma_{t+1}(j)}^{(t+1)}-\frac{s}{N}\right]_r\left[u_{\sigma_{t+1}(i)}^{(t+1)}-u_{\sigma_t(j)}^{(t)}+1-\frac{s}{N}\right]_r}{\left[u_{\sigma_t(i)}^{(t)}-u_{\sigma_t(j)}^{(t)}\right]_r\left[u_{\sigma_t(j)}^{(t)}-u_{\sigma_t(i)}^{(t)}-1\right]_r}.
\end{aligned}\tag{3.1}$$

Here meromorphic function $f_l(z_1^{(1)},\cdots,z_l^{(1)}|\cdots|z_1^{(N)},\cdots,z_l^{(N)})$ *is symmetric in each of variables* $(z_1^{(1)},\cdots,z_l^{(1)}),\cdots,(z_1^{(N)},\cdots,z_l^{(N)})$.
Let us set meromorphic function $(f_m*f_n)(z_1^{(1)},\cdots,z_{m+n}^{(1)}|\cdots|z_1^{(N)},\cdots,z_{m+n}^{(N)})$ *symmetric in each of variables* $(z_1^{(1)},\cdots,z_{m+n}^{(1)}),\cdots$ $(z_1^{(N)},\cdots,z_{m+n}^{(N)})$.

$$\begin{aligned}
&(f_m*f_n)(z_1^{(1)},\cdots,z_{m+n}^{(1)}|\cdots|z_1^{(N)},\cdots,z_{m+n}^{(N)})\\
=&\sum_{\sigma_1\in S_{m+n}}\sum_{\sigma_2\in S_{m+n}}\cdots\sum_{\sigma_N\in S_{m+n}}\\
\times&\ f_m(z_{\sigma_1(1)}^{(1)},\cdots,z_{\sigma_1(m)}^{(1)}|\cdots|z_{\sigma_N(1)}^{(N)},\cdots,z_{\sigma_N(m)}^{(N)})\\
\times&\ f_n(z_{\sigma_1(m+1)}^{(1)},\cdots,z_{\sigma_1(m+n)}^{(1)}|\cdots|z_{\sigma_N(m+1)}^{(N)},\cdots,z_{\sigma_N(m+n)}^{(N)})\\
\times&\prod_{t=1}^N\prod_{i=1}^m\prod_{j=m+1}^{m+n}\frac{\left[u_{\sigma_t(i)}^{(t)}-u_{\sigma_{t+1}(j)}^{(t+1)}+\frac{s}{N}\right]_{r-1}\left[u_{\sigma_{t+1}(i)}^{(t+1)}-u_{\sigma_t(j)}^{(t)}-1+\frac{s}{N}\right]_{r-1}}{\left[u_{\sigma_t(i)}^{(t)}-u_{\sigma_t(j)}^{(t)}\right]_{r-1}\left[u_{\sigma_t(j)}^{(t)}-u_{\sigma_t(i)}^{(t)}+1\right]_{r-1}}.
\end{aligned}\tag{3.2}$$

Here meromorphic function $f_l(z_1^{(1)},\cdots,z_l^{(1)}|\cdots|z_1^{(N)},\cdots,z_l^{(N)})$ *is symmetric in each of variables* $(z_1^{(1)},\cdots,z_l^{(1)}),\cdots,(z_1^{(N)},\cdots,z_l^{(N)})$.

We have a pair of infinitely many commutative family of Feigin-Odesskii algebra. Let us set theta function with parameters $\nu_1,\cdots,\nu_N$ and α.

$$\vartheta_{m,\alpha}(u_1^{(1)},\cdots,u_m^{(1)}|\cdots|u_1^{(N)},\cdots,u_m^{(N)})=\prod_{t=1}^N\left[\sum_{j=1}^m(u_j^{(t)}-u_j^{(t+1)})-\nu_t+\alpha\right]_r. \tag{3.3}$$

Proposition 3.2 *$\vartheta_{m,\vartheta}$ and $\vartheta_{n,\beta}$ commute each other with respect to the product (3.1).*

$$\vartheta_{m,\alpha} \circ \vartheta_{n,\beta} = \vartheta_{n,\beta} \circ \vartheta_{m,\alpha}. \tag{3.4}$$

Let us set theta function with parameters $\nu_1, \cdots, \nu_N$ and α.

$$\vartheta^*_{m,\alpha}(u_1^{(1)}, \cdots, u_m^{(1)} | \cdots | u_1^{(N)}, \cdots, u_m^{(N)}) = \prod_{t=1}^{N} \left[\sum_{j=1}^{m} (u_j^{(t+1)} - u_j^{(t)}) - \nu_t + \alpha \right]_{r-1}. \tag{3.5}$$

Proposition 3.3 *$\vartheta_{m,\alpha}$ and $\vartheta_{n,\beta}$ commute each other with respect to the product (3.2).*

$$\vartheta^*_{m,\alpha} * \vartheta^*_{n,\beta} = \vartheta^*_{n,\beta} * \vartheta^*_{m,\alpha}. \tag{3.6}$$

Proof of the above proposition is summarized in [10].

3.2 Free field realization

Let ϵ_j $(1 \leqq j \leqq N)$ be an orthonormal basis in $\mathbb{R}^N$ relative to the standard inner product $(\epsilon_i|\epsilon_j) = \delta_{i,j}$. Let us set $\bar{\epsilon}_j = \epsilon_j - \epsilon$ where $\epsilon = \frac{1}{N}\sum_{j=1}^N \epsilon_j$. We identify $\epsilon_{j+N} = \epsilon_j$. Let the weighted lattice $P = \sum_{j=1}^N \mathbb{Z}\bar{\epsilon}_j$. Let us set $\alpha_j = \bar{\epsilon}_j - \bar{\epsilon}_{j+1} \in P$. Let us introduce the bosons β_m^j $(m \in \mathbb{Z}_{\neq 0}; 1 \leqq j \leqq N)$ by

$$[\beta_m^i, \beta_n^j] = \begin{cases} m\frac{[(r-1)m]}{[rm]}\frac{[(s-1)m]}{[sm]}\delta_{m+n,0}, & (i = j) \\ -m\frac{[(r-1)m]}{[rm]}\frac{[m]}{[sm]}x^{sm\ \mathrm{sgn}(i-j)}\delta_{m,n}, & (i \neq j) \end{cases} \tag{3.7}$$

Let us set the commutation relations of P_λ, Q_μ $(\lambda, \mu \in P)$ by

$$[P_\lambda, iQ_\mu] = (\lambda|\mu). \tag{3.8}$$

We deal with the bosonic Fock space $\mathcal{F}_{l,k}, (l, k \in P)$ generated by $\beta^i_{-m}, (m > 0, i = 1, \cdots, N)$ over the vacuum vector $|l, k\rangle$.

$$\beta_m^i|l,k\rangle = 0 \ (m > 0, i = 1, \cdots, N), \tag{3.9}$$

$$P_\alpha|l,k\rangle = \left(\alpha \left| \sqrt{\frac{r}{(r-1)}}l - \sqrt{\frac{r-1}{r}}k \right.\right)|l,k\rangle, \tag{3.10}$$

$$|l,k\rangle = e^{\left(i\sqrt{\frac{r}{(r-1)}}Q_l - i\sqrt{\frac{r-1}{r}}Q_k\right)}|0,0\rangle. \tag{3.11}$$

Definition 3.4 *We set the screening currents $F_j(z), (1 \leqq j \leqq N)$ by*

$$F_j(z) \ = \ e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}}(x^{(\frac{2s}{N}-1)j}z)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j}+\frac{r-1}{r}}$$

$$\times \quad :\exp\left(\sum_{m\neq 0}\frac{1}{m}B_m^j z^{-m}\right):, \ (1\leqq j\leqq N-1) \tag{3.12}$$

$$F_N(z) = e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_N}}(x^{2s-N}z)^{\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_N}+\frac{r-1}{2r}}z^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}+\frac{r-1}{2r}}$$

$$\times \quad :\exp\left(\sum_{m\neq 0}\frac{1}{m}B_m^N z^{-m}\right):, \tag{3.13}$$

We set the screening currents $E_j(z),(1\leqq j\leqq N)$ by

$$E_j(z) = e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_j}}(x^{(\frac{2s}{N}-1)j}z)^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j}+\frac{r}{r-1}}$$

$$\times \quad :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]}{[(r-1)m]}B_m^j z^{-m}\right):, \ (1\leqq j\leqq N-1) \tag{3.14}$$

$$E_N(z) = e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_N}}(x^{2s-N}z)^{-\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_N}+\frac{r}{2(r-1)}}z^{\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_1}+\frac{r}{2(r-1)}}$$

$$\times \quad :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]}{[(r-1)m]}B_m^N z^{-m}\right):. \tag{3.15}$$

Here we have set

$$B_m^j = (\beta_m^j-\beta_m^{j+1})x^{-\frac{2s}{N}jm}, \ (1\leqq j\leqq N-1), \tag{3.16}$$

$$B_m^N = (x^{-2sm}\beta_m^N-\beta_m^1). \tag{3.17}$$

Proposition 3.5 *The currents $F_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ satisfy the following commutation relations.*

$$\left[u_1-u_2-\frac{s}{N}\right]_r F_j(z_1)F_{j+1}(z_2) = \left[u_2-u_1+\frac{s}{N}-1\right]_r F_{j+1}(z_2)F_j(z_1), \quad (1\leqq j\leqq N), \tag{3.18}$$

$$[u_1-u_2]_r[u_1-u_2+1]_rF_j(z_1)F_j(z_2) = [u_2-u_1]_r[u_2-u_1+1]_rF_j(z_2)F_j(z_1), \quad (1\leqq j\leqq N), \tag{3.19}$$

$$F_i(z_1)F_j(z_2) = F_j(z_2)F_i(z_1), \quad (|i-j|\geqq 2). \tag{3.20}$$

We read $F_{N+1}(z)=F_1(z)$. The currents $E_j(z)$, $(1\leqq j\leqq N; N\geqq 3)$ satisfy the following commutation relations.

$$\left[u_1-u_2+1-\frac{s}{N}\right]_{r-1} E_j(z_1)E_{j+1}(z_2) = \left[u_2-u_1+\frac{s}{N}\right]_{r-1} E_{j+1}(z_2)E_j(z_1), \quad (1\leqq j\leqq N), \tag{3.21}$$

$$[u_1-u_2]_{r-1}[u_1-u_2-1]_{r-1}E_j(z_1)E_j(z_2) = [u_2-u_1]_{r-1}[u_2-u_1-1]_{r-1}E_j(z_2)E_j(z_1), \quad (1\leqq j\leqq N), \tag{3.22}$$

$$E_i(z_1)E_j(z_2) = E_j(z_2)E_i(z_1), \quad (|i-j|\geqq 2). \tag{3.23}$$

We read $E_{N+1}(z)=E_1(z)$.

Proposition 3.6 *The screening currents* $E_j(z),F_j(z)$, $(1\leqq j\leqq N;N\geqq 3)$ *satisfy the following relation.*

$$[E_i(z_1),F_j(z_2)] = \frac{\delta_{i,j}}{x-x^{-1}}(\delta(xz_2/z_1)H_j(x^r z_2)-\delta(xz_1/z_2)H_j(x^{-r}z_2)),(1\leqq i,j\leqq N). \tag{3.24}$$

Here we have set

$$\begin{aligned} H_j(z) &= x^{(1-\frac{2s}{N})2j}e^{-\frac{i}{\sqrt{r(r-1)}}Q_{\alpha_j}}(x^{(\frac{2s}{N}-1)j}z)^{-\frac{1}{\sqrt{r(r-1)}}P_{\alpha_j}+\frac{1}{r(r-1)}} \\ &\times :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]}{[(r-1)m]}B_m^j z^{-m}\right):,\quad (1\leqq j\leqq N-1), \end{aligned} \tag{3.25}$$

$$\begin{aligned} H_N(z) &= x^{2(N-2s)}e^{-\frac{i}{\sqrt{r(r-1)}}Q_{\alpha_N}}(x^{2s-N}z)^{-\frac{1}{\sqrt{r(r-1)}}P_{\bar{\epsilon}_N}+\frac{1}{2r(r-1)}}z^{-\frac{1}{\sqrt{r(r-1)}}P_{\bar{\epsilon}_1}+\frac{1}{2r(r-1)}} \\ &\times :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]}{[(r-1)m]}B_m^N z^{-m}\right):. \end{aligned} \tag{3.26}$$

Definition 3.7 *Let us set the functional* $\mathcal{G}$ *by*

$$\begin{aligned} \mathcal{G}(f_m) &= \oint\cdots\oint\prod_{t=1}^N\prod_{j=1}^m\frac{dz_j^{(t)}}{2\pi i z_j^{(t)}}F_1(z_1^{(1)})\cdots F_1(z_m^{(1)})\cdots F_N(z_1^{(N)})\cdots F_N(z_m^{(N)}) \\ &\times \frac{\prod_{t=1}^N\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_r\left[u_j^{(t)}-u_i^{(t)}-1\right]_r}{\prod_{t=1}^{N-1}\prod_{i,j=1}^m\left[u_i^{(t)}-u_j^{(t+1)}+1-\frac{s}{N}\right]_r\prod_{i,j=1}^m\left[u_i^{(1)}-u_j^{(N)}+\frac{s}{N}\right]_r} \\ &\times f_m(z_1^{(1)},\cdots,z_m^{(1)}|\cdots|z_1^{(N)},\cdots,z_m^{(N)}). \end{aligned} \tag{3.27}$$

We take the integration contours to be simple closed curves around the origin satisfying

$$|x^{\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|,\ (1\leqq t\leqq N-1,1\leqq i,j\leqq m),$$
$$|x^{2-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{-\frac{2s}{N}}z_j^{(1)}|,\ (1\leqq i,j\leqq m).$$

Let us set the functional $\mathcal{G}^*$ *by followings.*

$$\begin{aligned} \mathcal{G}^*(f_m) &= \oint\cdots\oint\prod_{t=1}^N\prod_{j=1}^m\frac{dz_j^{(t)}}{2\pi i z_j^{(t)}}E_1(z_1^{(1)})\cdots E_1(z_m^{(1)})\cdots E_N(z_1^{(N)})\cdots E_N(z_m^{(N)}) \\ &\times \frac{\prod_{t=1}^N\prod_{1\leqq i<j\leqq m}\left[u_i^{(t)}-u_j^{(t)}\right]_{r-1}\left[u_j^{(t)}-u_i^{(t)}-1\right]_{r-1}}{\prod_{t=1}^{N-1}\prod_{i,j=1}^m\left[u_i^{(t)}-u_j^{(t+1)}-1+\frac{s}{N}\right]_{r-1}\prod_{i,j=1}^m\left[u_i^{(1)}-u_j^{(N)}-\frac{s}{N}\right]_{r-1}} \end{aligned}$$

$$\times \quad f_m(z_1^{(1)},\cdots,z_m^{(1)}|\cdots|z_1^{(N)},\cdots,z_m^{(N)}). \tag{3.28}$$

We take the integration contours to be simple closed curves around the origin satisfying

$$|x^{\frac{2s}{N}}z_j^{(t+1)}|<|z_i^{(t)}|<|x^{-2+\frac{2s}{N}}z_j^{(t+1)}|,\ (1\leqq t\leqq N-1, 1\leqq i,j\leqq m),$$
$$|x^{2-\frac{2s}{N}}z_j^{(1)}|<|z_i^{(N)}|<|x^{-\frac{2s}{N}}z_j^{(1)}|,\ (1\leqq i,j\leqq m).$$

Proposition 3.8 *When the functions $f_l(z_1^{(1)},\cdots,z_l^{(1)}|\cdots|z_1^{(N)},\cdots,z_l^{(N)})$ is meromorphic function symmetric in each of varibles $(z_1^{(t)},\cdots,z_l^{(t)})$, $(1\leqq t\leqq N)$, and don't have poles at the origin $z_j^{(t)}=0$, $(1\leqq t\leqq N, 1\leqq j\leqq l)$, the functionals $\mathcal{G}$, $\mathcal{G}^*$ satisfy*

$$\mathcal{G}(f_m)\mathcal{G}(f_n)=\mathcal{G}(f_m\circ f_n), \tag{3.29}$$

$$\mathcal{G}^*(f_m)\mathcal{G}^*(f_n)=\mathcal{G}^*(f_m * f_n). \tag{3.30}$$

Symmetrizing with respect to the integration variables $(z_1^{(t)},\cdots,z_{m+n}^{(t)})$ product $\mathcal{G}(f_m)\mathcal{G}(f_n)$, we have the above proposition.

In what follows we set parameters in the theta function $\vartheta_{m,\alpha}$, $\vartheta^*_{m,\alpha}$; $\nu_t=\sqrt{r(r-1)}P_{\bar{\epsilon}_{t+1}}$, $(1\leqq t\leqq N)$, $\alpha=\sum_{t=1}^N \alpha_t P_{\bar{\epsilon}_t}$, $(\alpha_t\in\mathbb{C})$. Because the relation $\sum_{t=1}^N P_{\bar{\epsilon}_t}=0$, $\vartheta_{m,\alpha}$, $\vartheta^*_{m,\alpha}$ have $(N-1)$ independent parameters.

Theorem 3.9 *For $r>1$ we have*

$$[\mathcal{G}(\vartheta_{m,\alpha}),\mathcal{G}(\vartheta_{n,\beta})]=0,\quad (m,n\in\mathbb{N}), \tag{3.31}$$

$$[\mathcal{G}^*(\vartheta^*_{m,\alpha}),\mathcal{G}^*(\vartheta^*_{n,\beta})]=0,\quad (m,n\in\mathbb{N}). \tag{3.32}$$

Theorem 3.10 *For $0<r<1$ we have*

$$[\mathcal{G}(\vartheta_{m,\alpha}),\mathcal{G}^*(\vartheta^*_{n,\beta})]=0,\quad (m,n\in\mathbb{N}). \tag{3.33}$$

Definition of $\mathcal{G}^*(\vartheta^*_{m,\alpha})$ for $0<r<1$ is given as the same manner as (4.44). See detailds in [10]. We have constructed infinitely many commutative operators $\mathcal{G}(\vartheta_{m,\alpha})$, $\mathcal{G}^*(\vartheta^*_{m,\alpha})$, $(m\in\mathbb{N})$ acting on the bosonic Fock space, which is regarded as the free field realization of commutative family of Feigin-Odesskii algebra (3.1) and (3.2).

4 Level k generalization of $U_{q,p}(\widehat{sl_2})$

In this section we consider level k generaliztion of section 2. Main contribution is construction of free field realization for one parameter s deformation of Level k elliptic albegra $U_{q,p}(\widehat{sl_2})$.

4.1 Feigin-Odesskii algebra

Let us set parameters $r,k \in \mathbb{R}$ such that $r > 0, r-k > 0$. It's not difficult to give Level k generalization of Feigin-Odesskii algebra: $f \circ g$ and $f * g$.

Definition 4.1 *Let us set the symmetric function* $(f_m \circ f_n)(z_1, \cdots, z_{m+n}|w_1, \cdots, w_{m+n})$ *by the same relation (2.3).*

Let us set the symmetric function $(f_m * f_n)(z_1 \cdots z_{m+n}|w_1 \cdots w_{m+n})$ *by modification of (2.4).*

$$\begin{aligned}
& (f_m * f_n)(z_1, \cdots, z_{m+n}|w_1, \cdots, w_{m+n}) \\
= & \frac{1}{((m+n)!)^2} \sum_{\sigma \in S_{m+n}} \sum_{\tau \in S_{m+n}} f_m(z_{\sigma(1)}, \cdots, z_{\sigma(m)}|w_{\tau(1)}, \cdots, w_{\tau(m)}) \\
& \times f_n(z_{\sigma(m+1)}, \cdots, z_{\sigma(m+n)}|w_{\tau(m+1)}, \cdots, w_{\tau(m+n)}) \\
& \times \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[v_{\tau(i)} - u_{\sigma(j)} - \frac{s}{2}\right]_{r-k} \left[u_{\sigma(i)} - v_{\tau(j)} - \frac{s}{2}\right]_{r-k}}{[u_{\sigma(i)} - u_{\sigma(j)}]_{r-k}[u_{\sigma(j)} - u_{\sigma(i)} + 1]_{r-k}} \\
\times & \prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \frac{\left[u_{\sigma(j)} - v_{\tau(i)} - \frac{s}{2} + 1\right]_{r-k} \left[v_{\tau(j)} - u_{\sigma(i)} - \frac{s}{2} + 1\right]_{r-k}}{[v_{\tau(j)} - v_{\tau(i)} + 1]_{r-k}[v_{\tau(j)} - v_{\tau(i)} + 1]_{r-k}}.
\end{aligned} \tag{4.1}$$

Here $f_l(z_1, \cdots, z_l|w_1, \cdots, w_l)$ *are meromorphic function symmetric in each of varibles* $(z_1, \cdots, z_l)$ *and* $(w_1, \cdots, w_l)$.

We have infinitely many commutative solutions $\vartheta_{m,\alpha}$ and $\vartheta^*_{m,\alpha}$ with respect with product $f \circ g$ and $f * g$. The solutions $\vartheta_{m,\alpha}(z_1, \cdots, z_m)$ for product $\circ$ is given as the same as (2.5). Let us set the theta function $\vartheta^*_{m,\alpha}$ with parmeters α, ν.

$$\vartheta_m(z_1, \cdots, z_m|w_1, \cdots, w_m) = \left[\sum_{j=1}^{m}(v_j - u_j) - \nu + \alpha\right]_{r-k} \left[\sum_{j=1}^{m}(u_j - v_j) - \alpha\right]_{r-k}. \tag{4.2}$$

Proposition 4.2 $\vartheta_{m,alpha}$ *and* $\vartheta_{n,\beta}$ *commute with respect to the product (4.1).*

$$\vartheta_{m,\alpha} * \vartheta_{n,\beta} = \vartheta_{n,\beta} * \vartheta_{m,\alpha}. \tag{4.3}$$

4.2 Free field realization

In this section we give one parameter deformation of Wakimoto realization of elliptic algebra $U_{q,p}(\widehat{sl_2})$ [2, 3]. Let us set deformation parameter $0 < s < 2$. Let us set the bosons $\alpha^j_m, \widetilde{\alpha}^j_m, (j = 1, 2; m \in \mathbb{Z}_{\neq 0})$,

$$[\alpha^j_m, \alpha^j_n] = -\frac{1}{m}\frac{[2m][rm]}{[km][(r-k)m]}\delta_{m+n,0}, \quad (j = 1, 2), \tag{4.4}$$

$$[\alpha_m^1,\alpha_n^2] = \frac{1}{m}\left(\frac{x^{(-r+k)m}([sm]-[(s-2)m])}{[(r-k)m]}+\frac{x^{km}([sm]+[(s-2)m])}{[km]}\right)\delta_{m+n,0}, \quad (4.5)$$

$$[\tilde{\alpha}_m^j,\tilde{\alpha}_n^j] = -\frac{1}{m}\frac{[2m][(r-k)m]}{[km][rm]}\delta_{m+n,0}, \quad (j=1,2), \quad (4.6)$$

$$[\tilde{\alpha}_m^1,\tilde{\alpha}_n^2] = \frac{1}{m}\left(\frac{x^{rm}(-[sm]+[(s-2)m])}{[rm]}+\frac{x^{km}([sm]+[(s-2)m])}{[km]}\right)\delta_{m+n,0}, \quad (4.7)$$

$$[\alpha_m^j,\tilde{\alpha}_n^j] = -\frac{1}{m}\frac{[2m]}{[km]}\delta_{m+n,0}, \quad (j=1,2), \quad (4.8)$$

$$[\alpha_m^1,\tilde{\alpha}_n^2] = \frac{1}{m}\frac{[sm]+[(s-2)m]}{[km]}\delta_{m+n,0}, \quad (4.9)$$

$$[\tilde{\alpha}_m^1,\alpha_n^2] = \frac{1}{m}\frac{[sm]+[(s-2)m]}{[km]}\delta_{m+n,0}. \quad (4.10)$$

We set the bosons β_m^j,γ_m^j, $(j=1,2;m\in\mathbb{Z}_{\neq 0})$,

$$[\beta_m^j,\beta_n^j] = \frac{[2m][(k+2)m]}{m}\delta_{m+n,0}, \ (j=1,2), \quad (4.11)$$

$$[\beta_m^1,\beta_n^2] = -\frac{[(k+2)m]([sm]+[(s-2)m])}{m}\delta_{m+n,0}, \quad (4.12)$$

$$[\gamma_m^j,\gamma_n^j] = \frac{1}{m}\frac{[2m]}{[km]}\delta_{m+n,0}, \quad (j=1,2), \quad (4.13)$$

$$[\gamma_m^1,\gamma_n^2] = -\frac{1}{m}\frac{[sm]+[(s-2)m]}{[km]}\delta_{m+n,0}. \quad (4.14)$$

We set the zero-mode operators $P_0,Q_0,\ h,\alpha$ and $h_0,h_1,h_2,\alpha_0,\alpha_1,\alpha_2$,

$$[P_0,iQ_0]=1,\ [h,\alpha]=2, \quad (4.15)$$

$$[h_0,\alpha_0]=[h_1,\alpha_2]=[h_2,\alpha_1]=(2-s), \quad [h_1,\alpha_1]=[h_2,\alpha_2]=0. \quad (4.16)$$

We set the Fock space $\mathcal{F}_{K,L}$, $(K,L\in\mathbb{Z})$.

$$\mathcal{F}_{K,L}=\bigoplus_{n,n_0,n_1,n_2\in\mathbb{Z}}\mathbb{C}[\alpha_{-m}^j,\tilde{\alpha}_{-m}^j,\beta_{-m}^j,\gamma_{-m}^j,(j=1,2;m\in\mathbb{N}_{\neq 0})]\otimes|K,L\rangle_{n,n_0,n_1,n_2}, \quad (4.17)$$

$$|K,L\rangle_{n,n_0,n_1,n_2}=e^{\left(L\sqrt{\frac{r}{2(r-k)}}-K\sqrt{\frac{r-k}{2r}}\right)iQ}\otimes e^{n\alpha}\otimes e^{n_0\alpha_0}\otimes e^{n_1\alpha_1}\otimes e^{n_2\alpha_2}. \quad (4.18)$$

Upon specialization $s\to 2$, simplification occures.

$$\alpha_m^2=-\alpha_m^1, \quad \tilde{\alpha}_m^1=\frac{[(r-k)m]}{[rm]}\alpha_m^1, \quad \tilde{\alpha}_m^2=-\frac{[(r-k)m]}{[rm]}\alpha_m^1, \quad (4.19)$$

$$\beta_m^2=-\beta_m^1, \quad \gamma_m^2=-\gamma_m^1, \quad h_0=h_1=h_2=\alpha_0=\alpha_1=\alpha_2=0. \quad (4.20)$$

The bosons $\alpha_m^1,\beta_m^1,\gamma_m^1$ are the same bosons which were introduced to construct the elliptic current associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ [2, 3, 4]. In order to construct infinitly many

commutative operators, we introduce one prameter s deformation of the bosons in [2, 3, 4]. We introduce the operators $C_j(z), C_j^\dagger(z)$, $(j=1,2)$ acting on the Fock space $\mathcal{F}_{J,K}$.

$$C_1(z) = e^{-\sqrt{\frac{2r}{k(r-k)}}iQ_0}e^{-\sqrt{\frac{2r}{k(r-k)}}P_0\log z}:\exp\left(-\sum_{m\neq 0}\alpha_m^1 z^{-m}\right):, \tag{4.21}$$

$$C_2(z) = e^{\sqrt{\frac{2r}{k(r-k)}}iQ_0}e^{\sqrt{\frac{2r}{k(r-k)}}P_0\log z}:\exp\left(-\sum_{m\neq 0}\alpha_m^2 z^{-m}\right):, \tag{4.22}$$

$$C_1^\dagger(z) = e^{\sqrt{\frac{2(r-k)}{kr}}iQ_0}e^{\sqrt{\frac{2(r-k)}{kr}}P_0\log z}:\exp\left(\sum_{m\neq 0}\widetilde{\alpha}_m^1 z^{-m}\right):, \tag{4.23}$$

$$C_2^\dagger(z) = e^{-\sqrt{\frac{2(r-k)}{kr}}iQ_0}e^{-\sqrt{\frac{2(r-k)}{kr}}P_0\log z}:\exp\left(\sum_{m\neq 0}\widetilde{\alpha}_m^2 z^{-m}\right):. \tag{4.24}$$

We set the operators $\widetilde{\Psi}_{j,I}(z), \widetilde{\Psi}_{j,II}(z), \widetilde{\Psi}_{j,I}^\dagger(z), \widetilde{\Psi}_{j,II}^\dagger(z)$, $(j=1,2)$ acting on the Fock space $\mathcal{F}_{J,K}$.

$$\begin{aligned}\widetilde{\Psi}_{j,I}(z) &= \exp\left(-(x-x^{-1})\sum_{m>0}\frac{x^{\frac{km}{2}}}{[m]_+}\beta_m^j z^{-m}\right) \\ &\times \exp\left(-\sum_{m>0}x^{-\frac{km}{2}}\gamma_{-m}^j z^m\right)\exp\left(-\sum_{m>0}x^{\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_m^j z^{-m}\right),\ (j=1,2),\end{aligned} \tag{4.25}$$

$$\begin{aligned}\widetilde{\Psi}_{j,II}(z) &= \exp\left((x-x^{-1})\sum_{m>0}\frac{x^{\frac{km}{2}}}{[m]_+}\beta_{-m}^j z^{m}\right) \\ &\times \exp\left(-\sum_{m>0}x^{\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_{-m}^j z^m\right)\exp\left(-\sum_{m>0}x^{-\frac{km}{2}}\gamma_m^j z^{-m}\right),\ (j=1,2),\end{aligned} \tag{4.26}$$

$$\begin{aligned}\widetilde{\Psi}_{j,I}^\dagger(z) &= \exp\left((x-x^{-1})\sum_{m>0}\frac{x^{-\frac{km}{2}}}{[m]_+}\beta_m^j z^{-m}\right) \\ &\times \exp\left(\sum_{m>0}x^{\frac{km}{2}}\gamma_{-m}^j z^m\right)\exp\left(\sum_{m>0}x^{-\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_m^j z^{-m}\right),\ (j=1,2),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\widetilde{\Psi}_{j,II}^\dagger(z) &= \exp\left(-(x-x^{-1})\sum_{m>0}\frac{x^{-\frac{km}{2}}}{[m]_+}\beta_{-m}^j z^{m}\right) \\ &\times \exp\left(\sum_{m>0}x^{-\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_{-m}^j z^m\right)\exp\left(\sum_{m>0}x^{\frac{km}{2}}\gamma_m^j z^{-m}\right),\ (j=1,2).\end{aligned} \tag{4.28}$$

We set the operators $\Psi_{j,I}(z), \Psi_{j,II}(z), \Psi_{j,I}^\dagger(z), \Psi_{j,II}^\dagger(z)$, $(j=1,2)$ acting on the Fock space $\mathcal{F}_{J,K}$.

$$\Psi_{1,I}(z) = \widetilde{\Psi}_{1,I}(z)e^{\alpha+\alpha_0+\alpha_1}x^{\frac{h}{2}+h_0+h_1}z^{-\frac{h}{k}}, \tag{4.29}$$

$$\Psi_{1,II}(z) = \widetilde{\Psi}_{1,II}(z)e^{\alpha+\alpha_0+\alpha_1}x^{-\frac{h}{2}+h_0-h_1}z^{-\frac{h}{k}}, \tag{4.30}$$

$$\Psi_{2,I}(z) = \widetilde{\Psi}_{2,I}(z)e^{-\alpha-\alpha_0+\alpha_2}x^{-\frac{h}{2}+h_0+h_2}z^{\frac{h}{k}}, \tag{4.31}$$

$$
\begin{aligned}
\Psi_{2,II}(z) &= \widetilde{\Psi}_{2,II}(z)e^{-\alpha-\alpha_0+\alpha_2}x^{\frac{h}{2}+h_0-h_2}z^{\frac{h}{k}}, && (4.32)\\
\Psi_{1,I}^{\dagger}(z) &= \widetilde{\Psi}_{1,I}^{\dagger}(z)e^{-\alpha-\alpha_0+\alpha_1}x^{\frac{h}{2}-h_0-h_1}z^{\frac{h}{k}}, && (4.33)\\
\Psi_{1,II}^{\dagger}(z) &= \widetilde{\Psi}_{1,II}^{\dagger}(z)e^{-\alpha-\alpha_0+\alpha_1}x^{-\frac{h}{2}-h_0+h_1}z^{\frac{h}{k}}, && (4.34)\\
\Psi_{2,I}^{\dagger}(z) &= \widetilde{\Psi}_{2,I}^{\dagger}(z)e^{\alpha+\alpha_0+\alpha_2}x^{-\frac{h}{2}-h_0-h_2}z^{-\frac{h}{k}}, && (4.35)\\
\Psi_{2,II}^{\dagger}(z) &= \widetilde{\Psi}_{2,II}^{\dagger}(z)e^{\alpha+\alpha_0+\alpha_2}x^{\frac{h}{2}-h_0+h_2}z^{-\frac{h}{k}}. && (4.36)
\end{aligned}
$$

Definition 4.3 *We set the operators $E_j(z), F_j(z)$, $(j=1,2)$, which can be regarded as one parameter deformation of the level k elliptic currents associated with the elliptic algebra $U_{q,p}(\widehat{sl_2})$ [3, 4].*

$$
E_j(z)=C_j(z)\Psi_j(z), \quad F_j(z)=C_j^{\dagger}(z)\Psi_j^{\dagger}(z), \quad (j=1,2), \tag{4.37}
$$

where we have set

$$
\Psi_j(z)=\frac{1}{x-x^{-1}}(\Psi_{j,I}(z)-\Psi_{j,II}(z)), \quad \Psi_j^{\dagger}(z)=\frac{-1}{x-x^{-1}}(\Psi_{j,I}^{\dagger}(z)-\Psi_{j,II}^{\dagger}(z)), \quad (j=1,2). \tag{4.38}
$$

Proposition 4.4 *The elliptic currents $E_j(z)$, $(j=1,2)$ satisfy the following commutation relations.*

$$
\begin{aligned}
& [u_1-u_2]_{r-k}[u_1-u_2-1]_{r-k}E_j(z_1)E_j(z_2) \\
=\; & [u_2-u_1]_{r-k}[u_2-u_1-1]_{r-k}E_j(z_2)E_j(z_1), \ (j=1,2), && (4.39)\\
& \left[u_1-u_2+\frac{s}{2}\right]_{r-k}\left[u_1-u_2-\frac{s}{2}+1\right]_{r-k}E_1(z_1)E_2(z_2) \\
=\; & \left[u_2-u_1+\frac{s}{2}\right]_{r-k}\left[u_2-u_1-\frac{s}{2}+1\right]_{r-k}E_2(z_2)E_1(z_1). && (4.40)
\end{aligned}
$$

The elliptic currents $F_j(z)$, $(j=1,2)$ satisfy the following commutation relations.

$$
\begin{aligned}
& [u_1-u_2]_{r}[u_1-u_2+1]_{r}F_j(z_1)F_j(z_2) \\
=\; & [u_2-u_1]_{r}[u_2-u_1+1]_{r}F_j(z_2)F_j(z_1), \ (j=1,2), && (4.41)\\
& \left[u_1-u_2-\frac{s}{2}\right]_{r}\left[u_1-u_2+\frac{s}{2}-1\right]_{r}F_1(z_1)F_2(z_2) \\
=\; & \left[u_2-u_1-\frac{s}{2}\right]_{r}\left[u_2-u_1+\frac{s}{2}-1\right]_{r}F_2(z_2)F_1(z_1). && (4.42)
\end{aligned}
$$

The currents $E_j(z)$ and $F_j(z)$ satisfy

$$
\begin{aligned}
[E_j(z_1),F_j(z_2)] &= \frac{x^{(-1)^j(s-2)}}{x-x^{-1}}\left(:C_j(z_1)C_j^{\dagger}(z_2)\Psi_{j,I}(z_1)\Psi_{j,I}^{\dagger}(z_2):\delta\left(\frac{x^kz_2}{z_1}\right)\right. && (4.43)\\
&- \left.:C_j(z_1)C_j^{\dagger}(z_2)\Psi_{j,II}(z_1)\Psi_{j,II}^{\dagger}(z_2):\delta\left(\frac{x^{-k}z_2}{z_1}\right)\right), \quad (j=1,2).
\end{aligned}
$$

Here we have used the delta-function $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$.

The definition of the functional $\mathcal{G}(f_m)$ is given as the same as (2.3).

Definition 4.5 *Let us set the functional $\mathcal{G}^*$ by followings.*

$$\begin{aligned}\mathcal{G}^*(f_m) &= \oint\prod_{j=1}^m \frac{dz_j}{2\pi i z_j}\oint\prod_{j=1}^m \frac{dw_j}{2\pi i w_j} E_1(z_1)\cdots E_1(z_m)E_2(w_1)\cdots E_2(w_m) \\ &\times \frac{\displaystyle\prod_{1\leqq j<k\leqq m}[u_i-u_j]_{r-1}[u_j-u_i+1]_{r-k}[v_i-v_j]_{r-1}[v_j-v_i+1]_{r-k}}{\displaystyle\prod_{i=1}^m\prod_{j=1}^m\left[u_i-v_j-\frac{s}{2}\right]_{r-k}\left[v_j-u_i-\frac{s}{2}+1\right]_{r-k}} f_m(z_1,\cdots,z_m|w_1,\cdots,w_m).\end{aligned} \tag{4.44}$$

We take the integration contours to be simple closed curves around the origin satisfying

$$|z_j^{(t)}|=1, \quad (t=1,2; j=1,2,\cdots,m).$$

Proposition 4.6 *When the function $f_l(z_1,\cdots,z_l|w_1,\cdots,w_l)$ are meromorphic function symmetric in each of varibles $(z_1,\cdots,z_l)$, $(w_1,\cdots,w_l)$, and don't have poles at the origin $z_j=0$, $w_j=0$, the functionals $\mathcal{G}$, $\mathcal{G}^*$ satisfy*

$$\mathcal{G}(f_m)\mathcal{G}(f_n)=\mathcal{G}(f_m\circ f_n), \tag{4.45}$$

$$\mathcal{G}^*(f_m)\mathcal{G}^*(f_n)=\mathcal{G}^*(f_m * f_n). \tag{4.46}$$

In what follows we set parameters $\nu=\sqrt{\frac{2r(r-k)}{k}}P_0+\frac{r-k}{r}h$ in theta function $\vartheta_{m,\alpha},\vartheta^*_{m,\alpha}$.

Theorem 4.7 *For $r>0$ and $r-k>0$, we have*

$$[\mathcal{G}(\vartheta_{m,\alpha}),\mathcal{G}(\vartheta_{n,\beta})]=0, \quad (m,n\in\mathbb{N}), \tag{4.47}$$

$$[\mathcal{G}^*(\vartheta^*_{m,\alpha}),\mathcal{G}^*(\vartheta^*_{n,\beta})]=0, \quad (m,n\in\mathbb{N}). \tag{4.48}$$

We have constructed infinitely many commutative operators $\mathcal{G}(\vartheta_{m,\alpha})$, $\mathcal{G}^*(\vartheta^*_{m,\alpha})$, $(m\in\mathbb{N})$ acting on the bosonic Fock space, which is regarded as the free field realization of commutative family of Feigin-Odesskii algebra (2.3) and (4.1).

5 Level k generalization of $U_{q,p}(\widehat{sl_N})$

In this section we report some results for Level k generalization of section 3, which are now in progress. Main result is free field realization of Level k elliptic algebra $U_{q,p}(\widehat{sl_N})$.

5.1 Feigin-Odesskii algebra

We introduce a pair of Feigin-Odesskii algebra.

Definition 5.1 *Let us set meromorphic function* $(f_m * f_n)(z_1^{(1)}, \cdots, z_{m+n}^{(1)}|\cdots|z_1^{(N)}, \cdots, z_{m+n}^{(N)})$ *symmetric in each of variables* $(z_1^{(1)}, \cdots, z_{m+n}^{(1)}), \cdots (z_1^{(N)}, \cdots, z_{m+n}^{(N)})$.

$$\begin{aligned} & (f_m * f_n)(z_1^{(1)}, \cdots, z_{m+n}^{(1)}|\cdots|z_1^{(N)}, \cdots, z_{m+n}^{(N)}) \\ = & \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \\ \times & f_m(z_{\sigma_1(1)}^{(1)}, \cdots, z_{\sigma_1(m)}^{(1)}|\cdots|z_{\sigma_N(1)}^{(N)}, \cdots, z_{\sigma_N(m)}^{(N)}) \\ \times & f_n(z_{\sigma_1(m+1)}^{(1)}, \cdots, z_{\sigma_1(m+n)}^{(1)}|\cdots|z_{\sigma_N(m+1)}^{(N)}, \cdots, z_{\sigma_N(m+n)}^{(N)}) \\ \times & \prod_{t=1}^N \prod_{i=1}^m \prod_{j=m+1}^{m+n} \frac{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_{t+1}(j)}^{(t+1)} + \frac{s}{N}\right]_{r-k} \left[u_{\sigma_{t+1}(i)}^{(t+1)} - u_{\sigma_t(j)}^{(t)} - 1 + \frac{s}{N}\right]_{r-k}}{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)}\right]_{r-k} \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} + 1\right]_{r-k}}. \end{aligned} \tag{5.1}$$

Here meromorphic function $f_l(z_1^{(1)}, \cdots, z_l^{(1)}|\cdots|z_1^{(N)}, \cdots, z_l^{(N)})$ *is symmetric in each of variables* $(z_1^{(1)}, \cdots, z_l^{(1)}), \cdots, (z_1^{(N)}, \cdots, z_l^{(N)})$.

The product $\circ$ is given by the same as (3.1). Let us set theta function with parameters $\nu_1, \cdots, \nu_N$ and α.

$$\vartheta_{m,\alpha}^*(u_1^{(1)}, \cdots, u_m^{(1)}|\cdots|u_1^{(N)}, \cdots, u_m^{(N)}) = \prod_{t=1}^N \left[\sum_{j=1}^m (u_j^{(t+1)} - u_j^{(t)}) - \nu_t + \alpha\right]_{r-k}. \tag{5.2}$$

Proposition 5.2 $\vartheta_{m,\alpha}$ *and* $\vartheta_{n,\beta}$ *commute each other with respect to the product (5.1).*

$$\vartheta_{m,\alpha}^* * \vartheta_{n,\beta}^* = \vartheta_{n,\beta}^* * \vartheta_{m,\alpha}^*. \tag{5.3}$$

5.2 Free field realization

In this section we give free field realization of Level k elliptic algebra $U_{q,p}(\widehat{sl_N})$. The author would like to emphasize that the free field realization of Level k is completely different from those of Level 1. We introduce free bosons $a_n^i, (1 \leqq i \leqq N-1; n \in \mathbb{Z}_{\neq 0})$, $b_n^{i,j}, (1 \leqq i < j \leqq N; n \in \mathbb{Z}_{\neq 0})$, $c_n^{i,j}, (1 \leqq i < j \leqq N; n \in \mathbb{Z}_{\neq 0})$, and the zero-mode operators $a^i, (1 \leqq i \leqq N-1)$, $b^{i,j}, (1 \leqq i < j \leqq N)$, $c^{i,j}, (1 \leqq i < j \leqq N)$.

$$[a_n^i, a_m^j] = \frac{[(k+N)n][A_{i,j}n]}{n}\delta_{n+m,0}, \quad [p_a^i, q_a^j] = (k+N)A_{i,j}, \tag{5.4}$$

$$[b_n^{i,j}, b_m^{k,l}] = -\frac{[n]^2}{n}\delta_{i,k}\delta_{j,l}\delta_{n+m,0}, \quad [p_b^{i,j}, q_b^{k,l}] = -\delta_{i,k}\delta_{k,l}, \tag{5.5}$$

$$[c_n^{i,j}, c_m^{k,l}] = \frac{[n]^2}{n}\delta_{i,k}\delta_{j,l}\delta_{n+m,0}, \quad [p_c^{i,j}, q_c^{k,l}] = \delta_{i,k}\delta_{j,l}. \tag{5.6}$$

Here the matrix $(A_{i,j})_{1 \leqq i,j \leqq N-1}$ represents the Cartan matrix of classical sl_N. For parameters $a_i \in \mathbb{R}, (1 \leqq i \leqq N-1)$, $b_{i,j} \in \mathbb{R}, (1 \leqq i < j \leqq N)$ $c_{i,j}, \in \mathbb{R}, (1 \leqq i < j \leqq N)$, we set the vacuum vector $|a,b,c\rangle$ of the Fock space $\mathcal{F}_{a,b,c}$ as following.

$$a_n^i|a,b,c\rangle = b_n^{j,k}|a,b,c\rangle = c_n^{j,k}|a,b,c\rangle = 0, \quad (n>0; 1 \leqq i \leqq N-1; 1 \leqq j < k \leqq N),$$

$$p_a^i|a,b,c\rangle = a_i|a,b,c\rangle, \ p_b^{j,k}|a,b,c\rangle = b_{j,k}|a,b,c\rangle, \ p_c^{j,k}|a,b,c\rangle = c_{j,k}|a,b,c\rangle, \quad (1 \leqq i \leqq N-1; 1 \leqq j < k \leqq N).$$

The Fock space $\mathcal{F}_{a,b,c}$ is generated by bosons $a_{-n}^i, b_{-n}^{j,k}, c_{-n}^{j,k}$ for $n \in \mathbb{N}_{\neq 0}$. The dual Fock space $\mathcal{F}_{a,b,c}^*$ is defined as the same manner. In this paper we construct the elliptic analogue of Drinfeld current for $U_{q,p}(\widehat{sl_N})$ by these bosons $a_n^i, b_n^{j,k}, c_n^{j,k}$ acting on the Fock space.

Let us set the bosonic operators $a_\pm^i(z), a^i(z), (1 \leqq i \leqq N-1)$, $b_\pm^{i,j}(z), b^{i,j}(z), c^{i,j}(z), (1 \leqq i < j \leqq N)$ by

$$a_\pm^i(z) = \pm(q-q^{-1})\sum_{n>0} a_{\pm n}^i z^{\mp n} \pm p_a^i \log q, \tag{5.7}$$

$$b_\pm^{i,j}(z) = \pm(q-q^{-1})\sum_{n>0} b_{\pm n}^{i,j} z^{\mp n} \pm p_b^{i,j} \log q, \tag{5.8}$$

$$a^i(z) = -\sum_{n \neq 0} \frac{a_n^i}{[(k+N)n]} q^{-\frac{k+N}{2}|n|} z^{-n} + \frac{1}{k+N}(q_a^i + p_a^i \log z), \tag{5.9}$$

$$b^{i,j}(z) = -\sum_{n \neq 0} \frac{b_n^{i,j}}{[n]} z^{-n} + q_b^{i,j} + p_b^{i,j} \log z, \tag{5.10}$$

$$c^{i,j}(z) = -\sum_{n \neq 0} \frac{c_n^{i,j}}{[n]} z^{-n} + q_c^{i,j} + p_c^{i,j} \log z, \tag{5.11}$$

Let us set the auxiliary operators $\gamma^{i,j}(z), \beta_1^{i,j}(z), \beta_2^{i,j}(z), \beta_3^{i,j}(z), \beta_4^{i,j}(z)$, $(1 \leqq i < j \leqq N)$ by

$$\gamma^{i,j}(z) = -\sum_{n \neq 0} \frac{(b+c)_n^{i,j}}{[n]} z^{-n} + (q_b^{i,j} + q_c^{i,j}) + (p_b^{i,j} + p_c^{i,j}) \log(-z), \tag{5.12}$$

$$\beta_1^{i,j}(z) = b_+^{i,j}(z) - (b^{i,j} + c^{i,j})(qz), \ \beta_2^{i,j}(z) = b_-^{i,j}(z) - (b^{i,j} + c^{i,j})(q^{-1}z), \tag{5.13}$$

$$\beta_3^{i,j}(z) = b_+^{i,j}(z) + (b^{i,j} + c^{i,j})(q^{-1}z), \ \beta_4^{i,j}(z) = b_-^{i,j}(z) + (b^{i,j} + c^{i,j})(qz). \tag{5.14}$$

We give a free field realization of Drinfeld current for $U_q(\widehat{sl_N})$.

Definition 5.1 *Let us set the bosonic operators $E^{\pm,i}(z), (1 \leqq i \leqq N-1)$ by*

$$E^{+,i}(z) = \frac{-1}{(q-q^{-1})z} \sum_{j=1}^{i} E_j^{+,i}(z), \tag{5.15}$$

$$E^{-,i}(z) = \frac{-1}{(q-q^{-1})z} \sum_{j=1}^{N-1} E_j^{-,i}(z), \tag{5.16}$$

where we have set

$$E_j^{+,i}(z) =: e^{\gamma^{j,i}(q^{j-1}z)}(e^{\beta_1^{j,i+1}(q^{j-1}z)} - e^{\beta_2^{j,i+1}(q^{j-1}z)})e^{\sum_{l=1}^{j-1}(b_+^{l,i+1}(q^{l-1}z)-b_+^{l,i}(q^l z))} :, \tag{5.17}$$

$$\begin{aligned} E_j^{-,i}(z) &= :e^{\gamma^{j,i+1}(q^{-(k+j)}z)}(e^{-\beta_4^{j,i}(q^{-(k+j)}z)} - e^{-\beta_3^{j,i}(q^{-(k+j)}z)}) \\ &\times \ e^{\sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-(k+l-1)}z)-b_-^{l,i}(q^{-(k+l)}z))+a_-^i(q^{-\frac{k+N}{2}}z)+\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-(k+l)}z)-b_-^{i+1,l}(q^{-(k+l-1)}z))} :, \\ &\text{for} \quad 1 \leqq j \leqq i-1, \end{aligned} \tag{5.18}$$

$$\begin{aligned} E_i^{-,i}(z) &= :e^{\gamma^{i,i+1}(q^{-(k+i)}z)+a_-^i(q^{-\frac{k+N}{2}}z)+\sum_{l=i+1}^{N}(b_-^{i,l}(q^{-(k+l)}z)-b_-^{i+1,l}(q^{-(k+l-1)}z))}: \\ &- \ :e^{\gamma^{i,i+1}(q^{k+i}z)+a_+^i(q^{\frac{k+N}{2}}z)+\sum_{l=i+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))} :, \end{aligned} \tag{5.19}$$

$$\begin{aligned} E_j^{-,i}(z) &= :e^{\gamma^{i,j+1}(q^{k+j}z)}(e^{\beta_2^{i+1,j+1}(q^{k+j}z)} - e^{\beta_1^{i+1,j+1}(q^{k+j}z)})e^{a_+^i(q^{\frac{k+N}{2}}z)+\sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z)-b_+^{i+1,l}(q^{k+l-1}z))} :, \\ &\text{for} \quad i+1 \leqq j \leqq N-1. \end{aligned} \tag{5.20}$$

Let us set the bosonic operators $\psi_i^{\pm}(z), (1 \leqq i \leqq N-1)$ *by*

$$\psi_{\pm}^i(q^{\pm\frac{k}{2}}z) =: e^{\sum_{j=1}^{i}(b_\pm^{j,i+1}(q^{\pm(k+j-1)}z)-b_\pm^{j,i}(q^{\pm(k+j)}z))+a_\pm^i(q^{\pm\frac{k+N}{2}}z)+\sum_{j=i+1}^{N}(b_\pm^{i,j}(q^{\pm(k+j)}z)-b_\pm^{i+1,j}(q^{\pm(k+j-1)}z))} : \tag{5.21}$$

Let us set

$$h_i = \sum_{j=1}^{i}(p_b^{j,i+1} - p_b^{j,i}) + p_a^i + \sum_{j=i+1}^{N}(p_b^{i,j} - p_b^{i+1,j}). \tag{5.22}$$

Let us introduce the auxiliary operators $\mathcal{B}_\pm^{*i,j}(z), \mathcal{B}_\pm^{i,j}(z)$, $(1 \leqq i < j \leqq N)$ by

$$\mathcal{B}_\pm^{*i,j}(z) = \exp\left(\pm\sum_{n>0}\frac{1}{[r^*n]}b_{-n}^{i,j}(q^{r^*-1}z)^n\right), \tag{5.23}$$

$$\mathcal{B}_\pm^{i,j}(z) = \exp\left(\pm\sum_{n>0}\frac{1}{[rn]}b_{n}^{i,j}(q^{-r^*+1}z)^{-n}\right). \tag{5.24}$$

Let us introduce the auxiliary operators $\mathcal{A}^{*i}(z), \mathcal{A}^i(z)$, $(1 \leqq i \leqq N-1)$ by

$$\mathcal{A}^{*i}(z) = \exp\left(\sum_{n>0}\frac{1}{[r^*n]}a_{-n}^i(q^{r^*}z)^n\right), \tag{5.25}$$

$$\mathcal{A}^{i}(z) = \exp\left(-\sum_{n>0}\frac{1}{[rn]}a_{n}^i(q^{-r^*}z)^{-n}\right). \tag{5.26}$$

Definition 5.2 *We define the dressing operators* $U^{*i}(z), U^i(z), (1 \leqq i \leqq N-1)$.

$$U^{*i}(z) = \left(\prod_{j=1}^{i-1}\mathcal{B}_+^{*j,i+1}(q^{2-j}z)\mathcal{B}_-^{*j,i}(q^{1-j}z)\right) \tag{5.27}$$

$$\times \quad \mathcal{B}_+^{*i,i+1}(q^{2-i}z)\mathcal{B}_+^{*i,i+1}(q^{-i}z)\left(\prod_{j=i+2}^{N}\mathcal{B}_+^{*i,j}(q^{-j+1}z)\mathcal{B}_-^{*i+1,j}(q^{-j+2}z)\right)\mathcal{A}^{*i}(q^{\frac{k-N}{2}}z),$$

$$U^i(z) = \left(\prod_{j=1}^{i-1}\mathcal{B}_-^{j,i+1}(q^{-2+j}z)\mathcal{B}_+^{j,i}(q^{-1+j}z)\right) \tag{5.28}$$

$$\times \quad \mathcal{B}_-^{i,i+1}(q^{-2+i}z)\mathcal{B}_-^{i,i+1}(q^{i}z)\left(\prod_{j=i+2}^{N}\mathcal{B}_-^{i,j}(q^{j-1}z)\mathcal{B}_+^{i+1,j}(q^{j-2}z)\right)\mathcal{A}^{i}(q^{\frac{-k+N}{2}}z).$$

Definition 5.3 *We define the elliptic deformation of Drinfeld current $E_i(z), F_i(z), H_i^{\pm}(z), (1 \leqq i \leqq N-1)$, by*

$$E_i(z) = U^{*i}(z)E^{+,i}(z)e^{2Q_i}z^{-\frac{P_i-1}{r-k}}, \tag{5.29}$$

$$F_i(z) = E^{-,i}(z)U^i(z)z^{\frac{h_i+P_i-1}{r}}, \tag{5.30}$$

$$H_i^+(z) = U^{*i}(q^{\frac{k}{2}}z)\psi_i^+(z)U^i(q^{-\frac{k}{2}}z)e^{2Q_i}q^{-h_i}(q^{(r-\frac{k}{2})}z)^{\frac{h_i+P_i-1}{r}-\frac{P_i-1}{r^*}}, \tag{5.31}$$

$$H_i^-(z) = U^{*i}(q^{-\frac{k}{2}}z)\psi_i^-(z)U^i(q^{\frac{k}{2}}z)e^{2Q_i}q^{h_i}(q^{-(r-\frac{k}{2})}z)^{\frac{h_i+P_i-1}{r}-\frac{P_i-1}{r^*}}. \tag{5.32}$$

Theorem 5.3 *The bosonic operators $E_i(z), F_i(z), H_i^{\pm}(z)$, $(1 \leqq i, j \leqq N-1)$ satisfy the following commutation relations.*

$$[u_1 - u_2 - \frac{A_{i,j}}{2}]_{r-k}E_i(z_1)E_j(z_2) = [u_1 - u_2 + \frac{A_{i,j}}{2}]_{r-k}E_j(z_2)E_i(z_1), \tag{5.33}$$

$$[u_1 - u_2 + \frac{A_{i,j}}{2}]_r F_i(z_1)F_j(z_2) = [u_1 - u_2 - \frac{A_{i,j}}{2}]_r F_j(z_2)F_i(z_1), \tag{5.34}$$

$$[E_i(z_1), F_j(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\left(\delta\left(q^{-k}\frac{z_1}{z_2}\right)H_i^+(q^{-\frac{k}{2}}z_1) - \delta\left(q^{k}\frac{z_1}{z_2}\right)H_i^-(q^{-\frac{k}{2}}z_2)\right). \tag{5.35}$$

We have constructed the free field realization of the elliptic algebra $U_{q,p}(\widehat{sl_N})$ for Level $k \neq 0, -N$. In order to construct free field realiztion of a pair of Feigin-Odesskii algebra (3.1) and (5.1), we have to solve the following problem.

Problem (1) Construct the free field realization of the currents $E_N(z)$ and $F_N(z)$, which satisfy the relations (5.33), (5.34) and (5.35) which are valid for all $1 \leqq i, j \leqq N$. (2) Construct one parameter s deformation of the free field realiztion of $E_j(z), F_j(z)$, $(1 \leqq j \leqq N)$.

After finishing the above problem, it is not difficult to construct the free field realization $\mathcal{G}$ and $\mathcal{G}^*$ of a pair of Feigin-Odesskii algebra (3.1) and (5.1).

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